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Nonregularizability of the Anisotropic Kepler Problem

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The anisotropic Kepler problem is a one parameter family of Hamiltonian systems recently introduced by Gutzwiller to approximate certain quantum mechanical systems. When the parameter $\mu = 1$, we have the ordinary Kepler or central force problem. This system is regularizable by any of several well known methods. When $\mu > 1$, the kinetic energy of the system becomes anisotropic. This destroys the integrability of the problem and changes the orbit structure of the system dramatically. In this paper, we show that the anisotropy of the kinetic energy also destroys the regularizability of the system, at least for most $\mu > 1$.

INTRODUCTION

The anisotropic Kepler problem is a one parameter family of classical mechanical systems with two degrees of freedom which depends analytically on a real parameter μ. When $\mu = 1$ the system reduces to the ordinary Kepler or Newtonian central force problem. This system is of course completely integrable and the orbit structure is well understood. When $\mu > 1$, some anisotropy is introduced into the system. This destroys the spherical symmetry of the system and changes the orbit structure dramatically.

The anisotropic Kepler problem was first introduced by Gutzwiller as a classical mechanical approximation to certain quantum mechanical systems. In particular, this system arises naturally when one looks for bound states of an electron near a donor impurity of a semiconductor. Here the potential is due to an ordinary Coulomb field, while the kinetic energy becomes anisotropic because of the electronic band structure in the solid. Gutzwiller [4] suggests that this situation is akin to an electron whose mass in one direction is larger than in the other directions. For more details on the quantum mechanical applications of this work, we refer the reader to [3]. We deal here only with the corresponding classical mechanical systems.

For $\mu = 1$, we have the ordinary (isotropic) Kepler problem. For negative total energy, orbits of this system are either closed or else lie on a cylinder of orbits which begin and end in collision with the origin. These latter orbits are

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singular solutions in the sense that they are not defined for all time. This situation can be rectified, however, by a process called regularization. Briefly, one extends such an orbit through collision via an "elastic bounce." That this can be done analytically has been known for a long time [10]. Thus the idea of regularization is to extend certain singular solutions so that they are defined for all time.

In recent years there have been other more global regularizations of the Kepler problem. We mention here the paper of Moser [9], which exhibits a conjugacy between the geodesic flow on a sphere and the regularized Kepler problem for negative energy. Also, Easton [2] has introduced a method of regularizing vector fields by surgery. It is this point of view that we shall adopt here: a summary of Easton's techniques is contained in Section 3.

The goal of this paper is to show that the anisotropic Kepler problem is non-regularizable in the sense of Easton, at least for most values of μ. This answers a question of Gutzwiller [4]. This also shows that the orbit structure of the anisotropic Kepler problem is much more complicated than the isotropic case. Indeed, in a separate paper [1], we have shown that for high anisotropy there exist subsystems of the anisotropic Kepler problem which are topologically conjugate to a Bernoulli shift on infinitely many symbols.

In Section 1 below, we outline the basic properties of the anisotropic Kepler problem. We show that, via a change of time scale, the singularity at the origin may be removed. In its place we attach a smooth boundary to each energy surface. The scaled flow then extends analytically over this boundary, which we call the collision manifold.

This technique has been exploited by McGehee in his study of triple collision in the collinear three body problem [7]. Indeed, the collision manifold he introduces at triple collision has many similarities to ours. In fact, triple collision represents a singularity which fails to be regularizable for exactly the same reasons that the anisotropic Kepler problem does.

In Section 2 below, we discuss the flow on the collision manifold in some detail. In Section 3 we then use this flow to prove the regularization result:

THEOREM. *There is an open and dense subset $\mathcal{O}$ of parameter values in* $(1, \infty)$ *for which the anisotropic Kepler problem cannot be regularized.*

The idea of the proof is to show that orbits which approach the origin close to the x-axis leave a neighborhood of the origin close to either the positive or negative y-axis. Hence nearby orbits tend to leave a neighborhood of collision far apart. It is for this reason that the singular collision orbits cannot be extended in any continuous fashion.

1. THE EQUATIONS OF MOTION

In this section we discuss the basic properties of the planar anisotropic Kepler problem. The configuration space for the system is $Q = R^2 - \{0\}$ with

Cartesian coordinates $\mathbf{q} = (q_1, q_2)$. The phase space is the tangent bundle to Q which we denote by TQ. We take coordinates $\mathbf{p} = (p_1, p_2)$ in each fiber.

The anisotropic Kepler problem is then given by a first order system of ordinary differential equations, or equivalently a vector field on TQ by

$$\begin{aligned}\dot{\mathbf{q}} &= M\mathbf{p},\\ \dot{\mathbf{p}} &= -\mathbf{q}/|\,\mathbf{q}\,|^3.\end{aligned} \tag{1.1}$$

Here M is the 2×2 matrix

$$M = \begin{pmatrix}\mu & 0\\ 0 & 1\end{pmatrix} \tag{1.2}$$

where $\mu \geqslant 1$. As we noted above, when $\mu = 1$, we have the ordinary Kepler problem.

Let X_μ denote the vector field on TQ given by (1.1). Note that X_μ has a singularity at $\mathbf{q} = 0$. Certain orbits of X_μ reach $\mathbf{q} = 0$ in finite time; these are the collision orbits of the system. The question of regularization is then whether or not these orbits can be extended through collision in any continuous sense.

The system (1.1) is a Hamiltonian system on TQ. Let V be the usual central force potential in the plane:

$$V = 1/|\,\mathbf{q}\,|. \tag{1.3}$$

Let K be the kinetic energy given by

$$K(\mathbf{p}) = \tfrac{1}{2}\mathbf{p}^t M\mathbf{p}. \tag{1.4}$$

Note that the potential energy remains spherically symmetric for all μ while the kinetic energy reflects the anisotropy of the system when $\mu > 1$. The total energy E of the system is then given by

$$E = K - V. \tag{1.5}$$

Then (1.1) may be written in Hamiltonian form with E as the Hamiltonian:

$$\begin{aligned}\dot{\mathbf{q}} &= \partial E/\partial\mathbf{p},\\ \dot{\mathbf{p}} &= -\partial E/\partial\mathbf{q}.\end{aligned} \tag{1.6}$$

Since (1.6) is Hamiltonian, the total energy E is an integral for the system, i.e., E is constant along solution curves of X_μ. Hence we may restrict attention to the invariant level sets of E. These are the so-called energy surfaces for the system. We henceforth consider the restriction of X_μ to a single such energy surface

$E^{-1}(e)$ which we denote by Σ_e. Σ_e is a three dimensional submanifold of $T\underset{\sim}{Q}$ since

$$\frac{\partial E}{\partial \mathbf{q}} = -\frac{\partial V}{\partial \mathbf{q}} = \frac{\mathbf{q}}{|\mathbf{q}|^3} \neq 0 \tag{1.7}$$

in $T\underset{\sim}{Q}$. Hence E has no critical points in $T\underset{\sim}{Q}$ and thus all of the level sets of E are smooth submanifolds of codimension one.

We wish to examine how the orbits of X_μ behave close to collision. To that end, we "blow up" the singularity at $\mathbf{q} = 0$ via a change of time scale. This has the effect of gluing an invariant boundary onto each Σ_e. The new flow extends analytically over this boundary, and we can "read off" the behavior of orbits close to collision from the phase portrait of the system on the boundary.

We first make a preliminary change of variables:

$$\begin{aligned} \mathbf{q} &= r\mathbf{s} \\ \mathbf{p} &= r^{-1/2}\mathbf{u} \end{aligned} \tag{1.8}$$

where $\mathbf{s}$ is a point on the unit circle S^1 and where $\mathbf{u}$ is a vector in R^2. In the new variables, the system (1.1) is transformed to

$$\begin{aligned} \dot{r} &= r^{-1/2}\mathbf{s}^t M\mathbf{u}, \\ \dot{\mathbf{s}} &= r^{-3/2}(M\mathbf{u} - (\mathbf{s}^t M\mathbf{u})\mathbf{s}), \\ \dot{\mathbf{u}} &= r^{-3/2}(\tfrac{1}{2}(\mathbf{s}^t M\mathbf{u})\mathbf{u} - \mathbf{s}), \end{aligned} \tag{1.9}$$

while the total energy relation becomes

$$\tfrac{1}{2}\mathbf{u}^t M\mathbf{u} = 1 + re. \tag{1.10}$$

The system (1.9) is an analytic vector field on the open manifold $(0, \infty) \times S^1 \times R^2$. Note that the set of collision points has been expanded to $\{0\} \times S^1 \times R^2$.

We now introduce a new time variable via

$$dt = r^{3/2}\, d\tau. \tag{1.11}$$

In the new time scale, the system (1.9) becomes

$$\begin{aligned} \dot{r} &= r(\mathbf{s}^t M\mathbf{u}), \\ \dot{\mathbf{s}} &= M\mathbf{u} - (\mathbf{s}^t M\mathbf{u})\mathbf{s}, \\ \dot{\mathbf{u}} &= \tfrac{1}{2}(\mathbf{s}^t M\mathbf{u})\mathbf{u} - \mathbf{s}, \end{aligned} \tag{1.12}$$

where the dot indicates differentiation with respect to τ.

We note several immediate consequences of this change of scale. First, (1.12) has no singularity at $r = 0$; in fact, this system extends analytically over all of $[0, \infty) \times S^1 \times R^2$. Secondly, the boundary $r = 0$ is now invariant under the flow. Thus this change of time scale has the effect of pasting an invariant boundary onto the phase space, and the other orbits of (1.9) are simply reparametrized. In the sequel we shall consider only the extended vector field (1.12), which we continue to denote by X_μ.

We now restrict attention to a single energy surface Σ_e. Using the energy relation (1.10), it follows that Σ_e meets the boundary $r = 0$ along the submanifold Λ defined by

$$\tfrac{1}{2}\mathbf{u}^t M \mathbf{u} = 1, \qquad \mathbf{s} \text{ arbitrary.} \tag{1.13}$$

Λ is clearly diffeomorphic to a two dimensional torus which we call the *collision manifold*. Note that Λ is independent of the total energy. Thus the change of time scale above also has the effect of pasting an invariant boundary onto each Σ_e. X_μ extends over this boundary as before, and is given by

$$\begin{aligned} \dot{\mathbf{s}} &= M\mathbf{u} - (\mathbf{s}^t M\mathbf{u})\mathbf{s}, \\ \dot{\mathbf{u}} &= \tfrac{1}{2}(\mathbf{s}^t M\mathbf{u})\mathbf{u} - \mathbf{s}. \end{aligned} \tag{1.14}$$

Orbits in Σ_e which previously began or ended in collision with the origin are slowed down by the change of time scale (1.11) and now tend asymptotically away from or toward Λ. Orbits which previously passed close to collision now come very close to Λ. How these orbits behave near the singularity is thus governed by the flow on Λ. We therefore discuss this flow in some detail in the next section.

2. The Flow on the Collision Manifold

The object of this section is to describe the flow given by (1.14) on the collision manifold. To facilitate the discussion, we introduce angular coordinates into phase space via

$$\begin{aligned} \mathbf{s} &= (\cos(\theta), \sin(\theta)), \\ \mathbf{u} &= (2(1 + re))^{1/2}(\mu^{-1/2}\cos(\psi), \sin(\psi)). \end{aligned} \tag{2.1}$$

The differential equations (1.12) become

$$\begin{aligned} \dot{r} &= 2r(1 + re)^{1/2}(\mu^{1/2}\cos(\psi)\cos(\theta) + \sin(\psi)\sin(\theta)), \\ \dot{\theta} &= 2(1 + re)^{1/2}(\sin(\psi)\cos(\theta) - \mu^{1/2}\cos(\psi)\sin(\theta)), \\ \dot{\psi} &= \frac{1}{(1 + re)^{1/2}}(\mu^{1/2}\sin(\psi)\cos(\theta) - \cos(\psi)\sin(\theta)). \end{aligned} \tag{2.2}$$

This system is again analytic in a neighborhood of $r = 0$, and on Λ, the system reduces to

$$\begin{aligned} \dot{\theta} &= 2(\sin(\psi)\cos(\theta) - \mu^{1/2}\cos(\psi)\sin(\theta)), \\ \dot{\psi} &= \mu^{1/2}\sin(\psi)\cos(\theta) - \cos(\psi)\sin(\theta). \end{aligned} \tag{2.3}$$

We use these and $(\mathbf{s}, \mathbf{u})$-coordinates interchangeably in the sequel.

Our first observation is that there are exactly eight equilibria for the system (2.3).

PROPOSITION 2.1. *The vector field X_μ admits exactly eight equilibrium points on Λ. The locations as well as the characteristic exponents of these equilibribria are as displayed in Table I.*

TABLE I

Equilibrium point	Characteristic exponents: On Λ	Characteristic exponents: Off Λ	Type
$(-\pi/2, -\pi/2)$	$-\frac{1}{2} \pm \frac{1}{2}(9 - 8\mu)^{1/2}$	2	Sink
$(0, 0)$	$-\mu^{1/2}/2 \pm \frac{1}{2}(9\mu - 8)^{1/2}$	$2\mu^{1/2}$	Saddle
$(\pi/2, \pi/2)$	$-\frac{1}{2} \pm \frac{1}{2}(9 - 8\mu)^{1/2}$	2	Sink
(π, π)	$-\mu^{1/2}/2 \pm \frac{1}{2}(9\mu - 8)^{1/2}$	$2\mu^{1/2}$	Saddle
$(-\pi/2, \pi/2)$	$\frac{1}{2} \pm \frac{1}{2}(9 - 8\mu)^{1/2}$	-2	Source
$(0, \pi)$	$\mu^{1/2}/2 \pm \frac{1}{2}(9\mu - 8)^{1/2}$	$-2\mu^{1/2}$	Saddle
$(\pi/2, -\pi/2)$	$\frac{1}{2} \pm \frac{1}{2}(9 - 8\mu)^{1/2}$	-2	Source
$(\pi, 0)$	$\mu^{1/2}/2 \pm \frac{1}{2}(9\mu - 8)^{1/2}$	$-2\mu^{1/2}$	Saddle

Proof. To see that these are the only equilibrium points on Λ we first note that $\dot{\theta} = 0$ iff both ψ and θ are multiples of π, or else

$$\cot(\theta) = \mu^{1/2}\cot(\psi).$$

On the other hand, $\dot{\psi} = 0$ iff both ψ and θ are multiples of π, or else

$$\cot(\psi) = \mu^{1/2}\cot(\theta).$$

Since $\mu > 1$, it follows that

$$\cot(\psi) = 0 = \cot(\theta).$$

Examination of these various possibilities then yields the result.

The computation of the characteristic exponents is straightforward, and is left to the reader. We simply note that

$$DX(p) = \begin{pmatrix} \nu & 0 \\ 0 & A \end{pmatrix}$$

where $\nu = \pm 2\mu^{1/2}$ or ± 2 depending on p, and where A is a 2×2 matrix giving the linearization of X_μ on Λ. Since

$$\mu < 9\mu - 8$$

it follows that

$$\frac{\mu^{1/2}}{2} < \frac{(9\mu - 8)^{1/2}}{2}$$

and hence that each of the saddle points in Λ has one positive and one negative eigenvalue, as required. Q.E.D.

As a consequence of Proposition 2.1, there are two sinks, two sources, and four hyperbolic saddle points in Λ for X_μ. Through each saddle point p there passes an invariant curve for the flow which consists of two orbits tending asymptotically toward p in forward time. This curve is called the stable manifold at p and is denoted by $W^s(p)$. Also, through each saddle point there passes another invariant curve consisting of two orbits backward asymptotic to p. This curve is the unstable manifold at p and is denoted by $W^u(p)$. The local behavior (near p) of each of these curves is well understood. Our goal for the remainder of this section is to understand how these curves behave far away from the saddle points.

Recall that a vector field on a manifold is called *gradient-like* if there exists a smooth real-valued function which increases along all nonequilibrium orbits.

PROPOSITION 2.2. *Let $f_\mu : \Lambda \to R$ be given by*

$$f_\mu(\mathbf{s}, \mathbf{u}) = |\, M^{-1/2}\mathbf{s}\,|^{-1/2}(\mathbf{s}^t\mathbf{u}) \tag{2.4}$$

where M^α is the 2×2 matrix given by

$$\begin{pmatrix} \mu^\alpha & 0 \\ 0 & 1 \end{pmatrix}.$$

Then X_μ is gradient-like with respect to f_μ.

Proof. We first compute the time derivative of f_μ along an orbit.

$$\begin{aligned} \dot{f}_\mu &= -\tfrac{1}{2}\,|\, M^{-1/2}\mathbf{s}\,|^{-5/2}(\mathbf{s}^t\mathbf{u})(\mathbf{s}^t M^{-1}\dot{\mathbf{s}}) + |\, M^{-1/2}\mathbf{s}\,|^{-1/2}(\dot{\mathbf{s}}^t\mathbf{u} + \mathbf{s}^t\dot{\mathbf{u}}) \\ &= |\, M^{-1/2}\mathbf{s}\,|^{-5/2}\{-\tfrac{1}{2}(\mathbf{s}^t\mathbf{u})^2 + \mathbf{s}^t M^{-1}\mathbf{s}(\mathbf{u}^t M\mathbf{u} - 1)\} \\ &= |\, M^{-1/2}\mathbf{s}\,|^{-5/2}\{-\tfrac{1}{2}(\mathbf{s}^t\mathbf{u})^2 + \mathbf{s}^t M^{-1}\mathbf{s}\}. \end{aligned}$$

Hence, using 2.1, we have

$$| M^{-1/2}s |^{5/2}\dot{f}_\mu = (\mu^{-1/2}\cos(\theta)\sin(\psi) - \sin(\theta)\cos(\psi))^2.$$

Hence $\dot{f}_\mu \geqslant 0$. Now if $\dot{f}_\mu = 0$, it follows from (2.2) that $\dot{\theta} = 0$. On the other hand, using 2.3, X_μ is never tangent to the curve defined by $\dot{\theta} = 0$ in Λ, except at at the equilibria. Hence f_μ increases along all non-equilibrium orbits. Q.E.D.

We remark that the fact that X_μ is gradient-like on Λ implies that there are no closed or recurrent orbits on Λ. In fact, the only non-wandering points for the restriction of X_μ to Λ are the eight equilibria above. It follows that all orbits must tend toward these equilibria in both forward and backward time. In particular, we have the following result concerning the stable and unstable manifolds of the saddle points.

PROPOSITION 2.3. *Each orbit in* $W^u(0, 0)$ *(resp.* $W^u(\pi, \pi)$*) is forward asymptotic to a distinct sink. Each orbit in* $W^s(0, \pi)$ *(resp.* $W^s(\pi, 0)$*) is backward asymptotic to a distinct source.*

Proof. Note first that the gradient function f_μ (2.4) achieves its maximum $2^{1/2}$ at the sinks in Λ, and its minimum $-2^{1/2}$ at the sources. Also observe that

$$\begin{aligned} f_\mu(0, 0) &= 2^{1/2}\mu^{1/4} = f_\mu(\pi, \pi) \\ f_\mu(\pi, 0) &= -2^{1/2}\mu^{-1/4} = f_\mu(0, \pi). \end{aligned} \tag{2.5}$$

Since f_μ must increase along nonequilibrium orbits, it follows that the unstable manifolds of (0, 0) and (π, π) fall directly into sinks, while the stable manifolds of $(\pi, 0)$ and $(0, \pi)$ emanate from sources. This completes the proof. Q.E.D.

Figure 1 gives a sketch of the phase portrait of the restriction of X_μ to Λ.

For the rest of this section, we confine our attention to the remaining invariant manifolds of the saddle points. There are two possibilities for the ultimate behavior of these curves. Either the unstable manifolds die in sinks and the stable manifolds emanate from sources as above, or else one or more of these curves match up and we have an orbit connecting two distinct saddles. Such an orbit is called a *saddle connection.* Our aim is to show that, for an open dense set of $\mu > 1$, there are no such saddle connections for X_μ.

We first consider the flow on the collision manifold when $\mu = 1$, i.e., for the ordinary Kepler problem. In this case, the differential equations on Λ are given by

$$\begin{aligned} \dot{\theta} &= 2\sin(\psi - \theta), \\ \dot{\psi} &= \sin(\psi - \theta). \end{aligned} \tag{2.6}$$

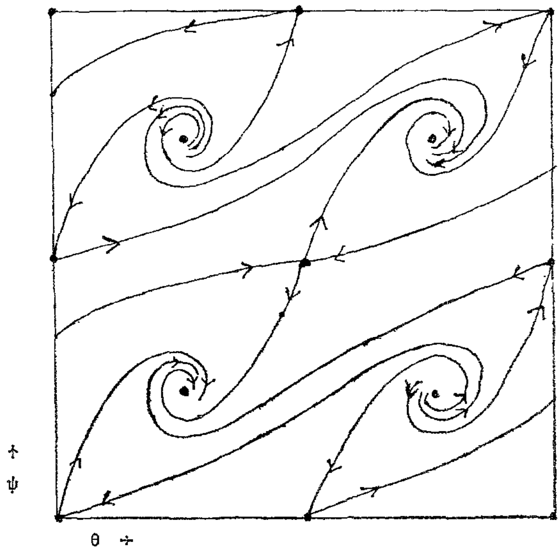

FIG. 1. The flow on the collision manifold Λ when $\mu > 1$.

Observe that there are two circles of equilibrium points for this system: one given by $\psi - \theta = 0$, the other by $\psi - \theta = \pi$. One computes readily that

$$DX_1(\theta, \theta) = \begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix} \tag{2.7}$$

which has eigenvalues 0 and -1, while

$$DX_1(\theta, \theta + \pi) = \begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix}$$

has eigenvalues 0 and $+1$. Hence the circle $\psi = \theta$ is an attractor for the flow, while $\psi = \theta + \pi$ is a repellor. Indeed, all nonequilibrium orbits of (2.6) are backward asymptotic to the circle $\psi = \theta + \pi$ and forward asymptotic to $\psi = \theta$. This can be seen as follows. From (2.6) we have

$$\dot{\psi} - \tfrac{1}{2}\dot{\theta} = 0.$$

Hence

$$\psi - \tfrac{1}{2}\theta = \text{constant} \tag{2.8}$$

along all orbits of (2.6). Thus the orbits of (2.6) are constrained to lie on the circles given by (2.8). Clearly, each such circle meets $\psi = \theta$ and $\psi = \theta + \pi$ in exactly one point, and one may check easily that the remaining orbits along such a circle travel between these two points as required.

Figure 2 gives a sketch of the phase portrait on the collision manifold when $\mu = 1$.

Note that the unstable manifold through the point $(-\pi, 0) = (\pi, 0)$ matches up exactly with the stable manifold through (π, π) when $\mu = 1$. For $\mu > 1$, both of these equilibria persist and are hyperbolic saddles. Furthermore, the stable and unstable manifolds through these points vary smoothly with μ. This follows since (2.7) implies that both circles of equilibria for the Kepler problem are normally hyperbolic. Hence their stable and unstable manifolds vary smoothly under perturbation. We refer to [6] for a proof of this fact.

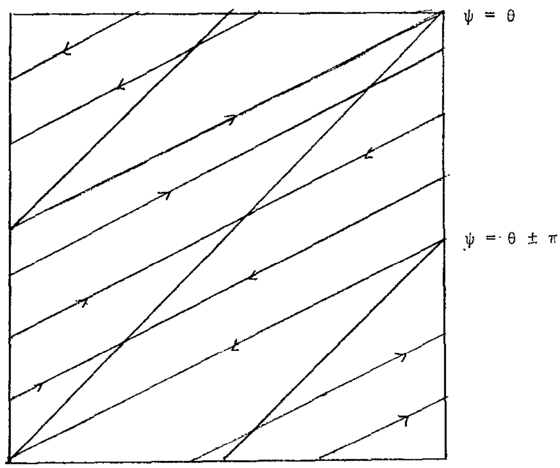

FIG. 2. The flow on the collision manifold Λ when $\mu = 1$ (the Kepler problem).

PROPOSITION 2.4. *For an open and dense set of $\mu > 1$, the stable manifold of X_μ at (π, π) does not match up with the unstable manifold at $(-\pi, 0)$.*

Proof. When $\mu = 1$, $W^u(-\pi, 0)$ matches up exactly with $W^s(\pi, \pi)$. By (2.8), one branch of $W^u(-\pi, 0)$ lies along the line

$$\psi = \tfrac{1}{2}\theta + \pi/2 \tag{2.9}$$

for $-\pi < \theta < \pi$. For μ close to 1, this branch of $W^u(-\pi, 0)$ varies only slightly. Let $\zeta(\theta, \mu)$ denote the ψ-coordinate of this branch, at least for $-\pi < \theta \leqslant 0$ and μ close to 1. So $\zeta(0, 1) = \pi/2$. Below we prove

LEMMA 2.5. $(\partial/\partial\mu)\, \zeta(0, 1) < 0$.

Consequently, $\zeta(0, \mu) < \pi/2$ for μ close to 1. Arguing similarly, one may also show that the $W^s(\pi, \pi)$ meets the line $\theta = 0$ slightly above the point $(0, \pi/2)$.

Now recall the gradient function (2.4). One computes easily that $f_\mu(0, \psi) > 0$

for $0 < \psi < \pi/2$ and that $f_\mu(0, \psi) < 0$ for $\pi/2 < \psi < \pi$. This implies that, for μ slightly larger than 1, the branches of $W^u(-\pi, 0)$ and $W^s(\pi, \pi)$ above do not match up.

The remaining branches of $W^u(-\pi, 0)$ and $W^s(\pi, \pi)$ also do not match up for μ close to 1, as one sees by applying the same argument along the line $\psi = \frac{1}{2}\theta - \pi/2$. Hence, using the gradient function above, we have

$$W^u(-\pi, 0) \cap W^s(\pi, \pi) = \phi$$

at least for μ close to 1.

Since X_μ varies analytically with μ, it follows [6] that $W^u(-\pi, 0)$ and $W^s(\pi, \pi)$ vary analytically with μ for $\mu > 1$. Hence these invariant manifolds can match up for at most a discrete set of parameter values $\mu > 1$. This completes the proof with the exception of Lemma 2.5.

Proof of Lemma 2.5. Eliminating time from (2.3) we have

$$\frac{d\psi}{d\theta} = \frac{\sin(\psi - \theta) + \epsilon \cos(\theta) \sin(\psi)}{2 \sin(\psi - \theta) - 2\epsilon \cos(\psi) \sin(\theta)} = F(\theta, \psi, \epsilon)$$

where we have written $\epsilon = \mu^{1/2} - 1$. In terms of ϵ, $\zeta(\theta, \epsilon)$ thus satisfies the equation

$$\zeta(\theta, \epsilon) = \int_{-\pi}^{\theta} F(s, \zeta(s), \epsilon)\, ds.$$

Let

$$\zeta(\theta, \epsilon) = \zeta_0(\theta) + \epsilon\zeta_1(\theta) + O(\epsilon^2).$$

We have shown (2.9) that

$$\zeta_0(\theta) = \tfrac{1}{2}\theta + \pi/2.$$

We now compute $\zeta_1(\theta)$:

$$\begin{aligned}
\zeta_1(\theta) &= \int_{-\pi}^{\theta} \left(\frac{\partial F}{\partial \epsilon} + \frac{\partial F}{\partial \psi}\, \zeta_1(s)\right) ds \\
&= \int_{-\pi}^{\theta} \frac{\sin(\zeta_0 + s)}{2 \sin(\zeta_0 - s)}\, ds \\
&= \frac{1}{2} \int_{-\pi}^{\theta} \frac{\sin(\frac{3}{2}s + \pi/2)}{\sin(-\frac{1}{2}s + \pi/2)}\, ds \\
&= \frac{1}{2} \int_{-\pi}^{\theta} \frac{\cos(\frac{3}{2}s)}{\cos(\frac{1}{2}s)}\, ds \\
&= \sin(\theta) - \pi/2 - \theta/2.
\end{aligned}$$

Thus, when $\theta = 0$, we have

$$\zeta_1(0) = \frac{-\pi}{2} = \frac{\partial}{\partial\epsilon}\,\zeta(0, 0).$$

This completes the proof of the lemma. Q.E.D.

Using similar techniques, it is not difficult to show:

PROPOSITION 2.6. *For an open and dense set of $\mu > 1$, each branch of $W^u(\pi, 0)$ and $W^u(0, \pi)$ misses all branches of $W^s(\pi, \pi)$ and $W^s(0, 0)$.*

Since the flow on Λ is gradient-like, the only other possibility for the ultimate behavior of the unstable manifolds above is that they die in sinks. Similarly, the stable manifolds above must emanate from sources. Combining these remarks with Proposition 2.3, we thus have

THEOREM 2.7. *There is an open and dense subset $\mathcal{O}$ of $(1, \infty)$ such that, if $\mu \in \mathcal{O}$, then the restriction of X_μ to Λ satisfies*

(a) *All of the stable manifolds of the saddle points emanate from the two sources.*

(b) *All of the unstable manifolds of the saddle points die in sinks.*

We wish to observe one final detail about the flow on Λ.

PROPOSITION 2.8. *Let $\mu \in \mathcal{O}$ and let p be one of the saddle points in Λ. Then each branch of $W^u(p)$ (resp. $W^s(p)$) dies in a distinct sink (resp. emanates from a distinct source).*

Proof. The differential equations (1.1) are invariant under the reflection

$$(\theta, \psi) \to (-\theta, -\psi). \tag{2.10}$$

Note that (2.10) fixes each saddle point and interchanges each pair of sinks (and each pair of sources). Also, one checks easily that (2.10) interchanges the two branches of the unstable manifold at p, and also the two branches of the stable manifold. Hence if one branch of $W^u(p)$ is asymptotic to the sink at $(\pi/2, \pi/2)$, then by symmetry, the other branch must be asymptotic to $(-\pi/2, -\pi/2)$. Similar arguments hold for $W^s(p)$, and this completes the proof. Q.E.D.

3. Non-regularizability of the Equations of Motion

In this section we consider the question of whether or not the anisotropic Kepler problem can be regularized on each energy surface. We adopt the topological point of view of Easton [2]; for connections between this type of regularization and the so-called analytic regularization we refer to [7].

We need some preliminary definitions. Let M be a C^∞ manifold and suppose X is a smooth vector field on M. We denote the (not necessarily complete) flow of X by ϕ_t, i.e., for any $p \in M$, $\phi_t(p)$ denotes the integral curve of X through p.

Let N be a submanifold with boundary of M satisfying $\dim N = \dim M$. We denote the boundary of N by n and distinguish three subsets of n:

$$\begin{aligned} n^+ &= \{p \in n \mid \text{for some } t < 0, \phi_s(p) \notin N, t < s < 0\}, \\ n^- &= \{p \in n \mid \text{for some } t > 0, \phi_s(p) \notin N, 0 < s < t\}, \\ \tau &= \{p \in n \mid X \text{ is tangent to } n \text{ at } p\}. \end{aligned}$$

These subsets are called the ingress, egress, and tangency sets in n respectively.

N is called an *isolating block* for X if

(a) τ is a codimension one submanifold of n,

(b) $\tau = n^+ \cap n^-$.

If N is an isolating block for X, then one checks easily that all tangencies of the vector field are exterior tangencies.

Our goal is to surround the set of singularities of the anisotropic Kepler problem with an isolating block. Thus we further define

$$\begin{aligned} a^+ &= \{p \in n \mid \phi_t(p) \in N \text{ for all } t \geqslant 0 \text{ for which } \phi_t(p) \text{ is defined}\}. \\ a^- &= \{p \in n \mid \phi_t(p) \in N \text{ for all } t \leqslant 0 \text{ for which } \phi_t(p) \text{ is defined}\}. \end{aligned}$$

Thus a^+ consists of all points in n whose forward orbits die in N, while a^- consists of those points whose backward orbits die in N. There is a natural Poincare map across the block

$$\Phi\colon n^+ - a^- \to n^- - a^-$$

obtained by following orbits through N until their first intersection with n^-. Φ is clearly a diffeomorphism. Finally, we say that X is regularizable (through N) if Φ extends to a homeomorphism of all of n^+. If the only singularities for X occur within N, then we say that the flow is regularizable. Intuitively, the flow is regularizable if one may connect all orbits which die at a singularity with an orbit that begins at the singularity in some continuous fashion.

In order to apply the above considerations to the anisotropic Kepler problem, we introduce the function $g\colon T\underset{\sim}{Q} \to R$ defined by

$$g(r, \mathbf{s}, \mathbf{u}) = r^2(\mathbf{s}^t M^{-1} \mathbf{s}).$$

Let S_ϵ denote the closed set $g \leqslant \epsilon$ in $T\underset{\sim}{Q}$.

PROPOSITION 3.1. *Suppose* $0 < \epsilon < \lambda/4e^2$ *where* $\lambda = \min(\mathbf{s}^t M^{-1}\mathbf{s})$. *Then* $S_\epsilon \cap \Sigma_e$ *is an isolating block for* X_μ *on the energy surface* Σ_e.

Proof. We first claim that the boundary $g = \epsilon$ defines a smooth torus in Σ_e. Clearly

$$r^2(\mathbf{s}^t M^{-1}\mathbf{s}) = \epsilon \tag{3.1}$$

defines a simple closed curve in the $(r, \mathbf{s})$-plane. Also, the energy relation

$$\tfrac{1}{2}\mathbf{u}^t M\mathbf{u} = 1 + er \tag{3.2}$$

defines an ellipse in the $\mathbf{u}$-plane, provided $1 + er > 0$. This is always true if $e \geqslant 0$. For $e < 0$ we have

$$r^2(\mathbf{s}^t M^{-1}\mathbf{s}) \leqslant \lambda/4e^2 \leqslant \frac{\mathbf{s}^t M^{-1}\mathbf{s}}{4e^2}$$

so that

$$r \leqslant -\tfrac{1}{2}e$$

along $g = e$. Hence $1 + er > 0$ in this case also. It follows that the boundary of S_ϵ is a smooth torus in Σ_e.

Next we investigate the behavior of orbits near the boundary $g = \epsilon$. Let $\dot{g}$ denote the time derivative of g along an orbit of X_μ. Using (1.14), one computes that

$$\begin{aligned} \dot{g} &= 2r\dot{r}(\mathbf{s}^t M^{-1}\mathbf{s}) + 2r^2(\mathbf{s}^t M^{-1}\dot{\mathbf{s}}) \\ &= 2r^2(\mathbf{s}^t\mathbf{u}) \end{aligned}$$

Hence $\dot{g} = 0$ iff $\mathbf{s}^t\mathbf{u} = 0$. If $\mathbf{s}^t\mathbf{u} = 0$, however, we have

$$\begin{aligned} \ddot{g} &= 4r\dot{r}(\mathbf{s}^t\mathbf{u}) + 2r^2(\dot{\mathbf{s}}^t\mathbf{u} + \mathbf{s}^t\dot{\mathbf{u}}) \\ &= 2r^2(\mathbf{u}^t M\mathbf{u} - 1) \\ &= 2r^2(1 + 2re) \end{aligned}$$

where we have used (1.10). Hence $\ddot{g} > 0$ if $e \geqslant 0$. If $e < 0$,

$$1 + 2re > 0$$

as we showed above. Hence g has a minimum along any orbit tangent to the boundary of S_ϵ. Finally, since $\mathbf{s}^t\mathbf{u} = 0$ defines two smooth circles in $g = \epsilon$, it follows that the tangency set is a smooth submanifold having codimension one. This completes the proof. Q.E.D.

As a corollary, we observe that any orbit which remains in S_ϵ for all $t \geqslant 0$ must be asymptotic to Λ. Indeed, if this is not the case, then the ω-limit set of such an orbit would lie entirely within S_ϵ. Since this set is compact and invariant, g would necessarily have a maximum along some orbit. But this cannot happen, as we showed in the proof of Proposition 3.1. Thus we have

COROLLARY 3.2. *If $x \in a^+$, then the orbit of x is forward asymptotic to Λ. Similarly, if $y \in a^-$, then the backward orbit of y is asymptotic to Λ.*

We now consider a particular orbit which tends asymptotically to Λ. Let γ denote the orbit of X_μ which is everywhere tangent to the positive q_1-axis and which tends toward Λ as $t \to \infty$. Such an orbit exists since the system (1.1) is invariant under the reflection

$$(q_1, q_2, p_1, p_2) \to (q_1, -q_2, p_1, -p_2).$$

Also, we denote by σ^+ (resp. σ^-) the orbit of X_μ which emanates from the origin and is everywhere tangent to the positive (resp. negative) q_2-axis.

PROPOSITION 3.3. (a) *γ is contained in the two dimensional stable manifold of $(0, 0, \pi)$ in Λ.*

(b) *σ^+ (resp. σ^-) is contained in the one dimensional unstable manifold of $(0, \pi/2, \pi/2)$ (resp. $(0, -\pi/2, -\pi/2)$).*

Proof. Any orbit which tends asymptotically (in forward or backward time) to Λ must tend toward one of the equilibria in Λ, since the flow of X_μ restricted to Λ is gradient-like. The result then follows immediately from the change of variables (2.1) together with Table I. Q.E.D.

Now the orbit γ cuts the isolating block S_ϵ at a point which we denote by p. Similarly, $\sigma^\pm$ meet S_ϵ at $q^\pm$. We wish to examine the behavior of the Poincare map Φ in a neighborhood of p, at least for the non-exceptional values of μ given by Theorem 2.7.

PROPOSITION 3.4. *If $\mu \in \mathcal{O}$, the Poincare map Φ for X_μ satisfies: for any neighborhoods U of p and $V^\pm$ of $q^\pm$ there exist points $u^+, u^- \in U$ such that $\Phi(u^+) \in V^+$ and $\Phi(u^-) \in V^-$.*

Proof. For ϵ small enough, $W^s(0, 0, \pi)$ meets S_ϵ transversely in a smooth curve. Let β: $[-1, 1] \to S_\epsilon$ be a smooth curve which meets $W^s(0, 0, \pi) \cap S_\epsilon$ transversely at $p = \beta(0)$. Then p divides this curve into two pieces, one on each "side" of $W^s(0, 0, \pi)$ for $|s|$ small. Let $\beta_i(s)$, $i = 1, 2$, denote these curves. We claim that there exists $\delta > 0$ such that, if $0 < |s| < \delta$, then $\Phi(\beta_1(s))$ is contained in one of the neighborhoods $V^\pm$, and $\Phi(\beta_2(s))$ is contained in the other.

Now $W^u(0, 0, \pi)$ consists of two branches: α^+, which dies in the sink at $(0, \pi/2, \pi/2)$, and α^-, asymptotic to $(0, -\pi/2, -\pi/2)$, at least for $\mu \in \mathcal{O}$. Let $W^\pm$ be small transversals to $\alpha^\pm$ respectively. By choosing $W^\pm$ smaller if necessary, we may assume that the forward orbits of all points in $W^+ - \Lambda$ leave S_ϵ through V^+, and similarly, the forward orbits of all points in $W^- - \Lambda$ exit through V^-.

Also, using standard arguments, it follows that for δ small enough, the forward orbit of each point in $\beta_1(s)$, $0 < |s| < \delta$, crosses one of W^+ or W^- before leaving S_ϵ. For definiteness, assume $\beta_1(s)$ crosses W^+. Then, for $|s|$ small enough, $\beta_2(s)$ crosses W^- before exiting S_ϵ. Hence $\Phi(\beta_1(s)) \subset V^+$ and $\Phi(\beta_2(s)) \subset V^-$, at least for $|s|$ small. This completes the proof. Q.E.D.

The proof above shows that points close to p are mapped by Φ to points close to either e^+ or e^-, depending on which "side" of $W^s(0, 0, \pi)$ the original points lie. Thus nearby pairs are mapped far apart by Φ. This implies that Φ cannot be extended continuously to p and thus shows that the flow of X_μ cannot be regularized. We finally remark that the flow on Λ is independent of total energy, and hence this result holds for all energy surfaces simultaneously.

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